Fast Optical System Identification by Numerical Interferometry

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Imaging through scattering media

- Numerous applications require imaging through scattering media:
  - Reconstructing scenes through fog
  - Imaging through tissues in the human body
  - Detecting patterns, cracks and material properties behind paint
  - Optical neural network backpropagation

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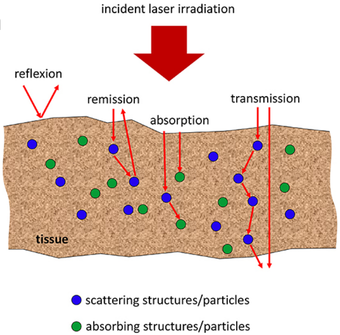

Wieneke S, Gerhard C. *Tissue optics and laser–tissue interactions.*

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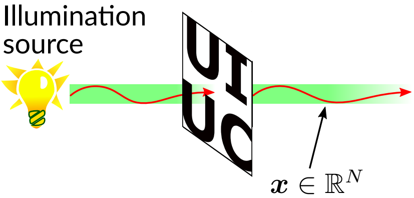
- Challenging physical limitations makes imaging in these scenarios prohibitively time consuming and expensive

Challenges in the imaging process

Illumination
source

Challenges in the imaging process



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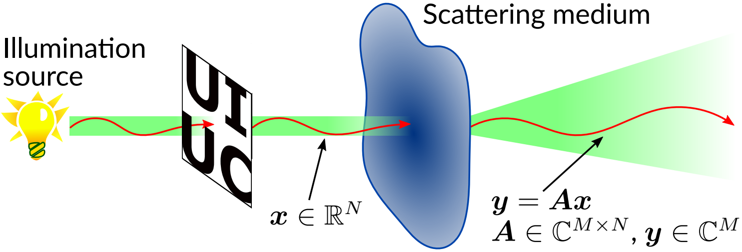

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- Get x by simple linear inversion?

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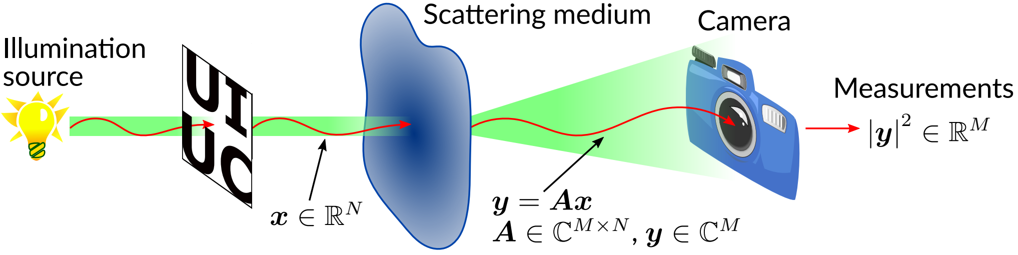

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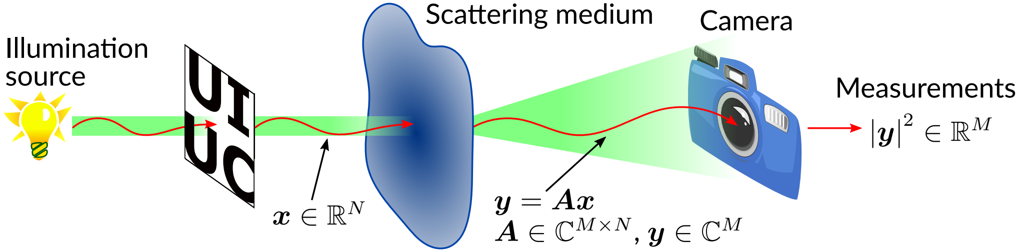

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Goal: Rapidly learn A

The way forward

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2. Learn $\boldsymbol{A} \in \mathbb{C}^{M\times N}$:

Double phase retrieval [1,2]

- Solve M quadratic equations separately, $|(\boldsymbol{y}^m)^*|^2 = |\boldsymbol{\Xi}^*(\boldsymbol{a}^m)^*|^2$, to recover each row
- 3.3 hours with GPU when $\boldsymbol{A}$ is $256^2 \times 64^2$

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- 3.3 hours with GPU when $\boldsymbol{A}$ is $256^2 \times 64^2$

NEW: Measurement phase retrieval

- Recover $\boldsymbol{Y}$ without knowing $\boldsymbol{A}$ [3]
  - Solve $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{\Xi}$ to recover $\boldsymbol{A}$
- 6.2 minutes with CPU when $\boldsymbol{A}$ is $256^2 \times 64^2$

3. Measure $|\boldsymbol{y}|^2 = |\boldsymbol{A}\boldsymbol{x}|^2$ for signal of interest, $\boldsymbol{x}$
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[3] One method shown in Gupta S et al. *Don't take it lightly: Phasing optical random projections with unknown operators.* NeurIPS 2019.

A linear system to recover transmission matrices

- With $\boldsymbol{Y}$ recovered and $\boldsymbol{\Xi}$ designed, instead of $|\boldsymbol{Y}|^2 = |\boldsymbol{A}\boldsymbol{\Xi}|^2$, solve

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{\Xi} \qquad \text{with} \qquad \boldsymbol{\Xi} \in \mathbb{R}^{N\times K}, \boldsymbol{A} \in \mathbb{C}^{M\times N}, \boldsymbol{Y} \in \mathbb{C}^{M\times K}$$

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- Design $\boldsymbol{\Xi}$ with full row rank and more probe signals, K, than N

- Least-squares fit $\widehat{\boldsymbol{A}} = \arg\min_{\boldsymbol{A}} \ \|\boldsymbol{Y} - \boldsymbol{A}\boldsymbol{\Xi}\|_F^2 = \boldsymbol{Y}\boldsymbol{\Xi}^\dagger$

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- Efficient $\boldsymbol{Y\Xi}^\dagger$:
    - Design $\boldsymbol{\Xi}$ as a concatenation of two circulant $N \times N$ matrices, $\boldsymbol{\Xi} = [\boldsymbol{\Xi}_A,\ \boldsymbol{\Xi}_B] \in \mathbb{R}^{N \times 2N}$
    - Use FFT to efficiently compute $\boldsymbol{Y\Xi}^\dagger$ as outlined in our paper

Fast transmission matrix identification

- We compute $\boldsymbol{A} \in \mathbb{C}^{M \times N}$ from real noisy optical hardware measurements:
  1. Imprint signals onto a coherent light beam
  2. Shines them through a multiple scattering medium, $\boldsymbol{A}$
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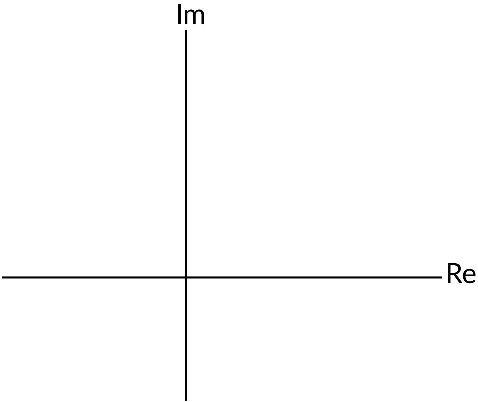
Time taken for measurement phase retrieval and solving $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{\Xi}$:

$\boldsymbol{N}$	$\boldsymbol{M/N}$	**Time (minutes)**
32^2	32	0.97
32^2	64	2.05
32^2	128	4.01
64^2	**16**	**6.15**
64^2	32	11.69
64^2	64	24.14
96^2	16	31.36
128^2	12	71.97

Trilateration to find the phase of a single measurement

- Optical measurement
  $\mathsf{y}_{10}^2 = |\langle \boldsymbol{a}, \xi_1 \rangle|^2 = |y_1|^2 \in \mathbb{R}$
  is squared distance to origin



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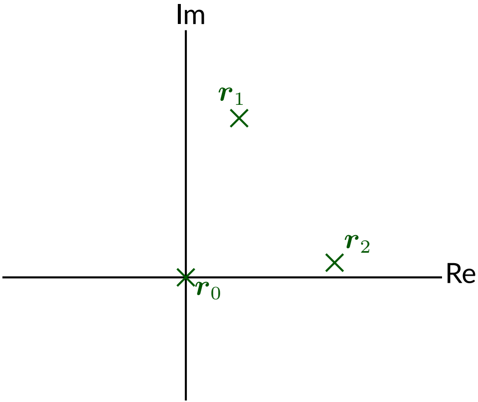

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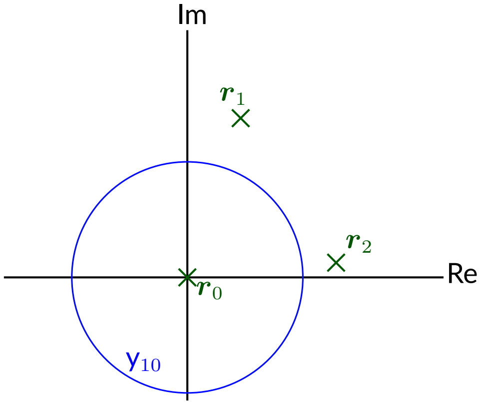

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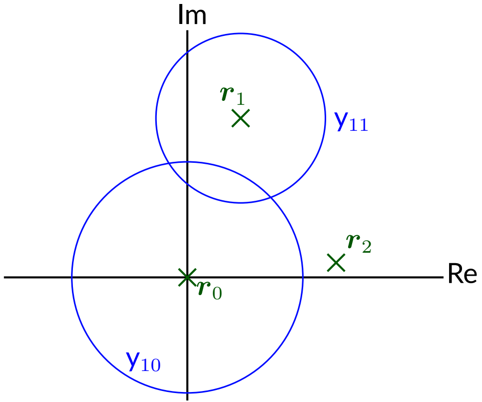

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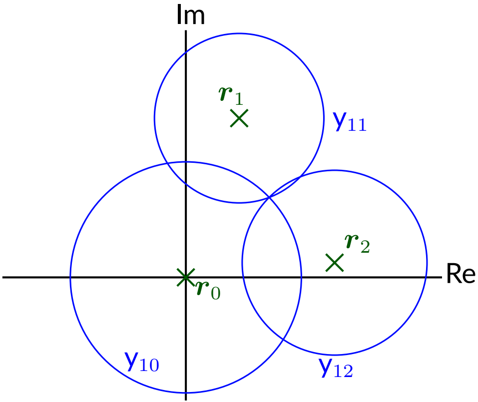

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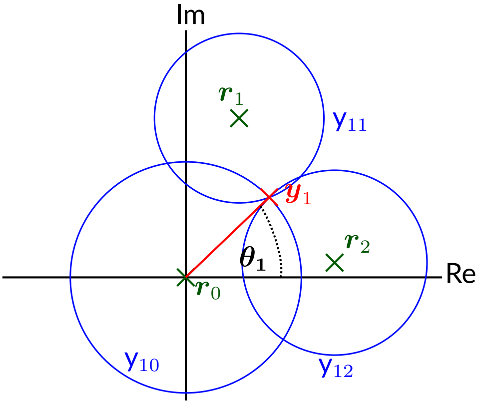

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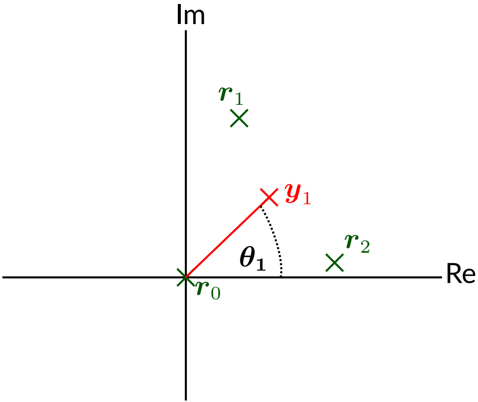

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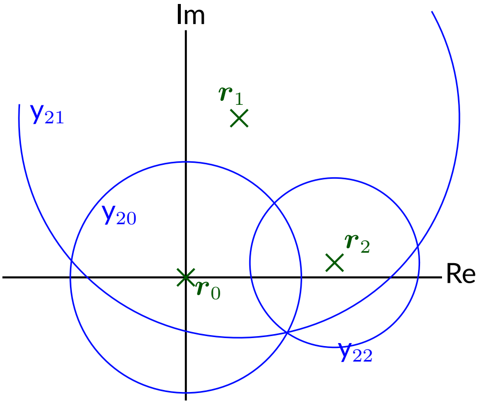

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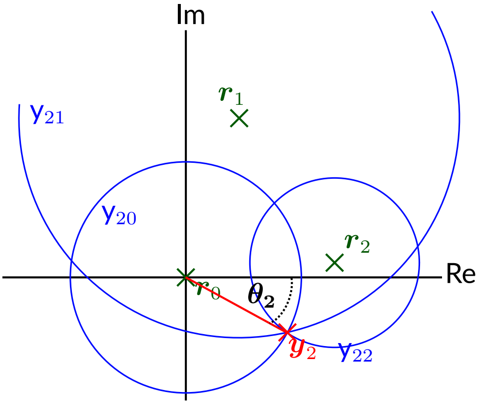

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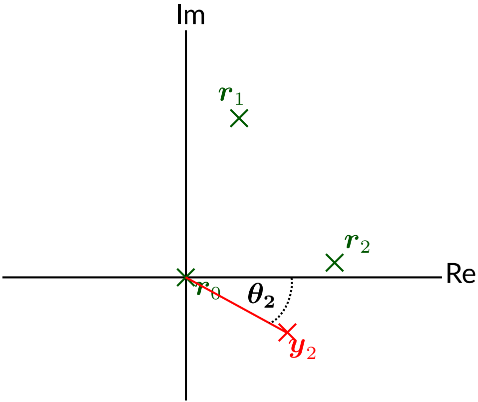

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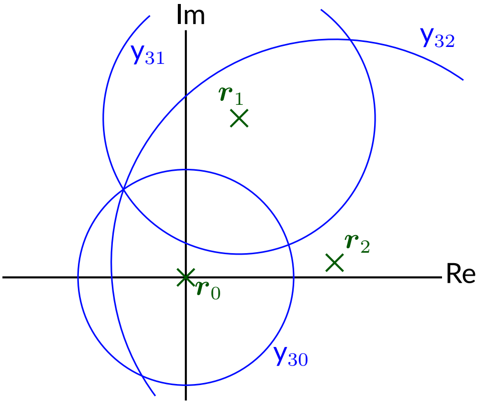

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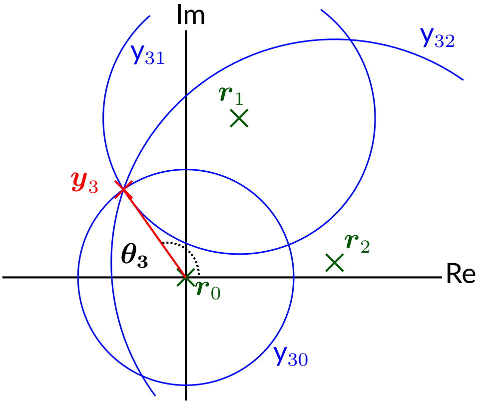

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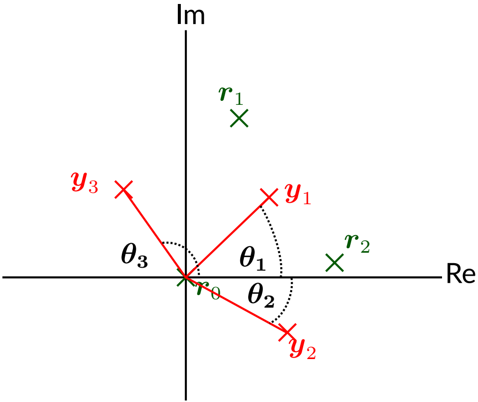

- **Can we measure distances to known points to perform measurement phase retrieval?**

Rapid numerical interferometry

- K known calibration signals: $\boldsymbol{\Xi} = [\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_K] \in \mathbb{R}^{N \times K}$
- $S \geq 3$ known anchor signals: $\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_S] \in \mathbb{R}^{N \times S}$
- For each row $\boldsymbol{a} \in \mathbb{C}^N$ of $\boldsymbol{A} \in \mathbb{C}^{M \times N}$:

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- Assume known:
  - $r_s := \langle \boldsymbol{a}, \boldsymbol{v}_s \rangle \in \mathbb{C}$
  - $\mathsf{r}_s := |r_s|$
- Unknowns:
  - $y_k := \langle \boldsymbol{a}, \boldsymbol{\xi}_k \rangle \in \mathbb{C}$
  - $\mathsf{y}_k := |y_k|$

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- $S \geq 3$ known anchor signals: $\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_S] \in \mathbb{R}^{N \times S}$
- For each row $\boldsymbol{a} \in \mathbb{C}^N$ of $\boldsymbol{A} \in \mathbb{C}^{M \times N}$:

- Assume known:
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- Unknowns:
  - $y_k := \langle \boldsymbol{a}, \boldsymbol{\xi}_k \rangle \in \mathbb{C}$
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- Measure:
  - $\mathsf{y}_{ks}^2 := |\langle \boldsymbol{a}, \boldsymbol{\xi}_k - \boldsymbol{v}_s \rangle|^2 = |y_k - r_s|^2$

Numerical interferometry rather than *optical* interferometry for signal interference

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$$\mathsf{y}_{ks}^2 = \mathsf{r}_s^2 + \mathsf{y}_k^2 - 2 r_s^T y_k$$

(Interpreting complex numbers as vectors in $\mathbb{R}^2$)

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$$\mathsf{y}_{ks}^2 - \mathsf{r}_s^2 = \left[-2 r_s^T, \ 1\right] \begin{bmatrix} y_k \\ \mathsf{y}_k^2 \end{bmatrix}$$

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$$\begin{bmatrix} \mathsf{y}_{k1}^2 - \mathsf{r}_1^2 \\ \vdots \\ \mathsf{y}_{kS}^2 - \mathsf{r}_S^2 \end{bmatrix} = \begin{bmatrix} -2r_1^T & 1 \\ \vdots & \vdots \\ -2r_S^T & 1 \end{bmatrix} \begin{bmatrix} y_k \\ \mathsf{y}_k^2 \end{bmatrix}$$

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$$\underbrace{\begin{bmatrix} \mathsf{y}_{11}^2 - \mathsf{r}_1^2 & \cdots & \mathsf{y}_{K1}^2 - \mathsf{r}_1^2 \\ \vdots & \vdots & \vdots \\ \mathsf{y}_{1S}^2 - \mathsf{r}_S^2 & \cdots & \mathsf{y}_{KS}^2 - \mathsf{r}_S^2 \end{bmatrix}}_{\boldsymbol{E} \in \mathbb{R}^{S \times K}} = \underbrace{\begin{bmatrix} -2r_1^T & 1 \\ \vdots & \vdots \\ -2r_S^T & 1 \end{bmatrix}}_{\boldsymbol{M} \in \mathbb{R}^{S \times 3}} \underbrace{\begin{bmatrix} y_1 & \cdots & y_K \\ \mathsf{y}_1^2 & \cdots & \mathsf{y}_K^2 \end{bmatrix}}_{\boldsymbol{W} \in \mathbb{R}^{3 \times K}}$$

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$$\boldsymbol{W} = \begin{bmatrix} y_1 & \cdots & y_K \\ \mathsf{y}_1^2 & \cdots & \mathsf{y}_K^2 \end{bmatrix} \qquad \underbrace{\boldsymbol{E}}_{\mathbb{R}^{S \times K}} = \underbrace{\boldsymbol{M}}_{\mathbb{R}^{S \times 3}} \underbrace{\boldsymbol{W}}_{\mathbb{R}^{3 \times K}} \qquad \Longrightarrow \qquad \widehat{\boldsymbol{W}} = \boldsymbol{M}^\dagger \boldsymbol{E}$$

Top two rows of $\widehat{W}$ are real and imaginary parts of $(a^* \, \Xi) \in \mathbb{C}^{1 \times K}$

Repeat for all rows of A and obtain Y in $Y = A\Xi$ without knowing A!

Getting initial anchor positions

- For row $\boldsymbol{a}$ of $\boldsymbol{A}$ for all (q, s) measure squared distances between anchor points on the complex plane: $|\langle \boldsymbol{a}, \boldsymbol{v}_q - \boldsymbol{v}_s \rangle|^2 = |r_q - r_s|^2$

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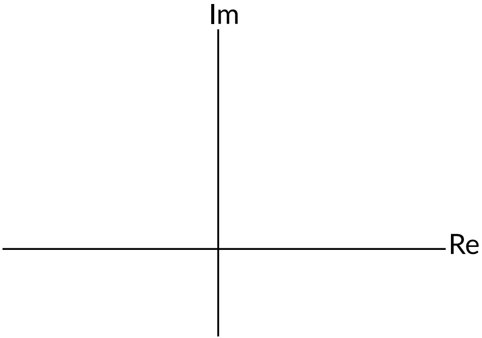
- Find a realization of points on complex plane satisfying distances
  - Can be done using multidimensional scaling (MDS)



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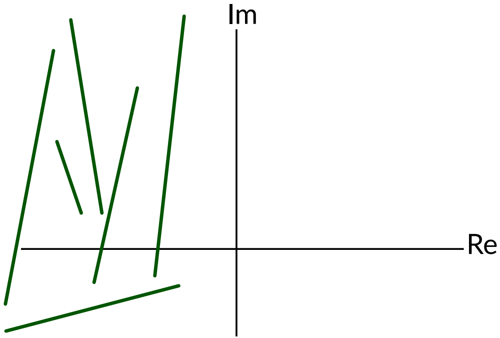

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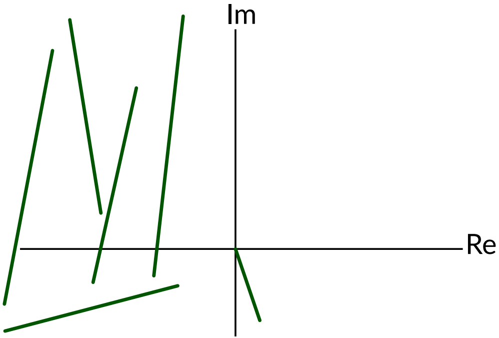

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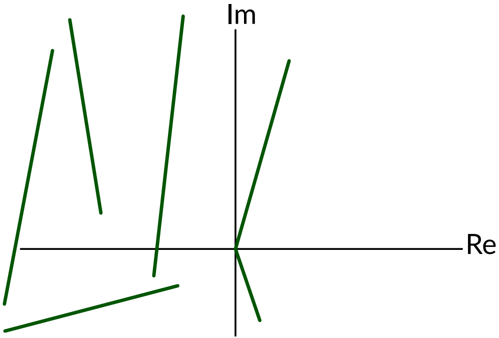

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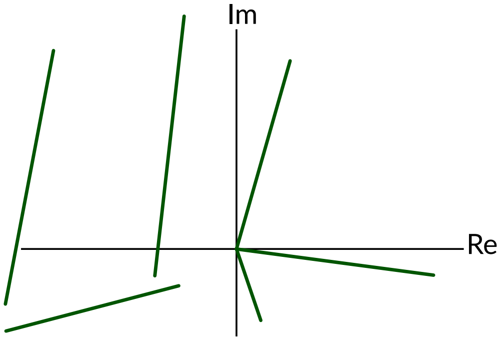

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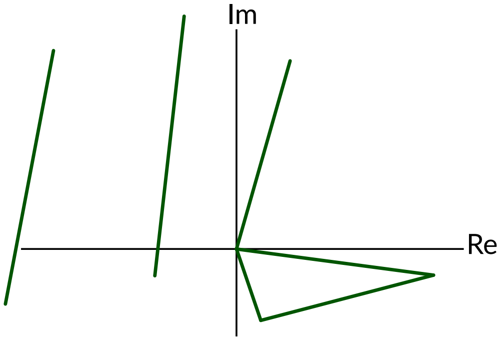

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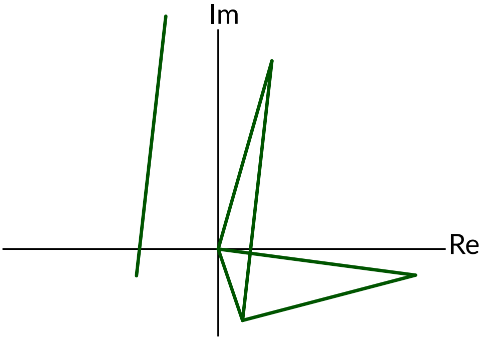

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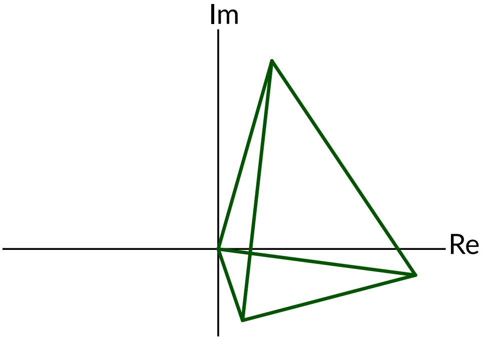

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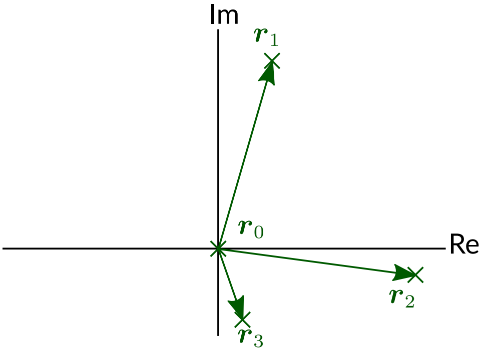
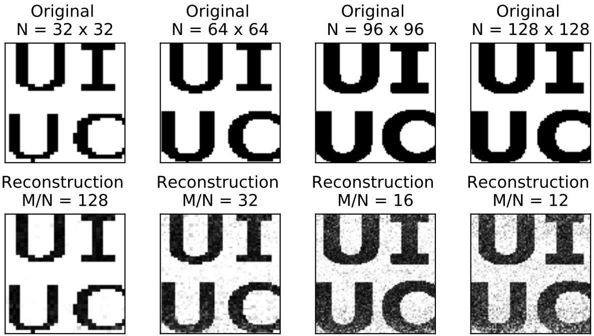


Experimental verification on optical hardware

- Wirtinger flow algorithm to reconstruct image, x, from optical measurements, $|Ax|^2$, using learned $A \in \mathbb{C}^{M \times N}$



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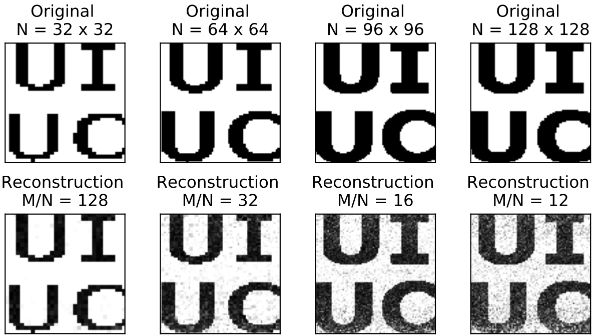
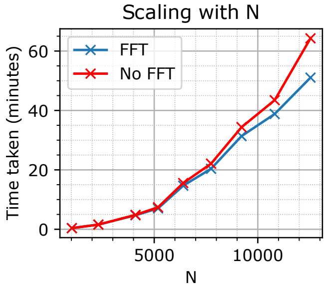


- Using the FFT method to solve $Y = A\Xi$ for A is more efficient as signal dimension increases



Summary

- Numerical interferometery enables rapid measurement phase retrieval
- Learning transmission matrices is a **linear problem instead of a quadratic one** with measurement phase retrieval
- **6.2 minutes** vs. 3.3 hours
- **Even with noisy optical measurements**, transmission matrices can be learned and used for imaging

Check out our paper for more details and link to code